How does a mathematician think about space and shape?

Università di Torino

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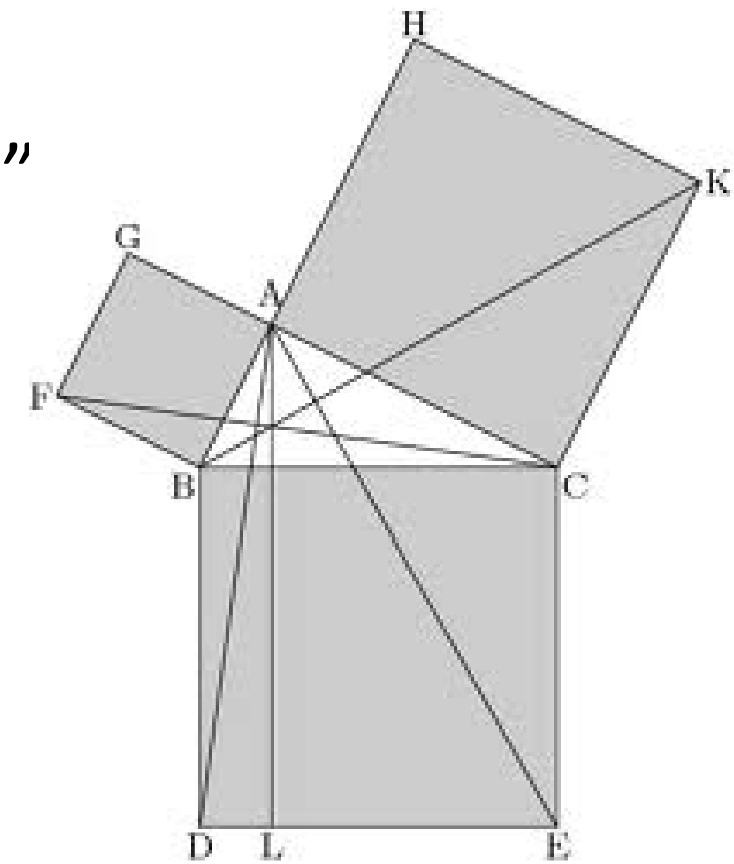
It all began with Euclid

- (although very similar ideas were developed earlier by the Babylonians, Indians and Chinese)
- Especially “Pythagoras’s” Theorem:
- Not just an obscure fact about triangles but the key to measuring distances in space



Euclid's towering influence

Matteo Ricci translates Euclid into Chinese (1605). This is from his Preface:

"At my university, I above all got to know one name: Euclid. He brought mathematical theory to great perfection, and he towers high above his predecessors. He has opened new avenues and has enlightened the path for later generations. … In the books he wrote … there is not a single thing that can be doubted. Especially his Elements is very exact and can rightly be called a standard work. … Everything is contained in his theory and there is nothing that does not follow from it."

The space of Euclid's Geometry seemed something absolute

Kant: Euclidean space as an *a priori*, not discovered but an unavoidable systematic framework for organizing our experiences

- "Geometry is a science which determines the properties of space synthetically, and yet *a priori*. … It must be originally intuition … found in the mind a priori, that is, before any perception of objects." (Critique of Pure Reason, Ch.I ,Part I, §3)
- Examples: that is three dimensional; that the sum of angles of a triangle equals two right angles.

In the early 19th century, mathematicians threw off the philosopher’s chains; in the early 20th, so did the physicists

- The cautious German Gauss, the swashbuckling Austro-Hungarian Bolyai, the pioneering Russian Lobachevsky all hit on the idea that Pythagoras’s theorem **didn’t have to hold** in the real world
- Einstein found that indeed **it doesn’t hold**, and your GPS allows for its failure. The space we live in has a complex geometry.

The fully abstract perspective

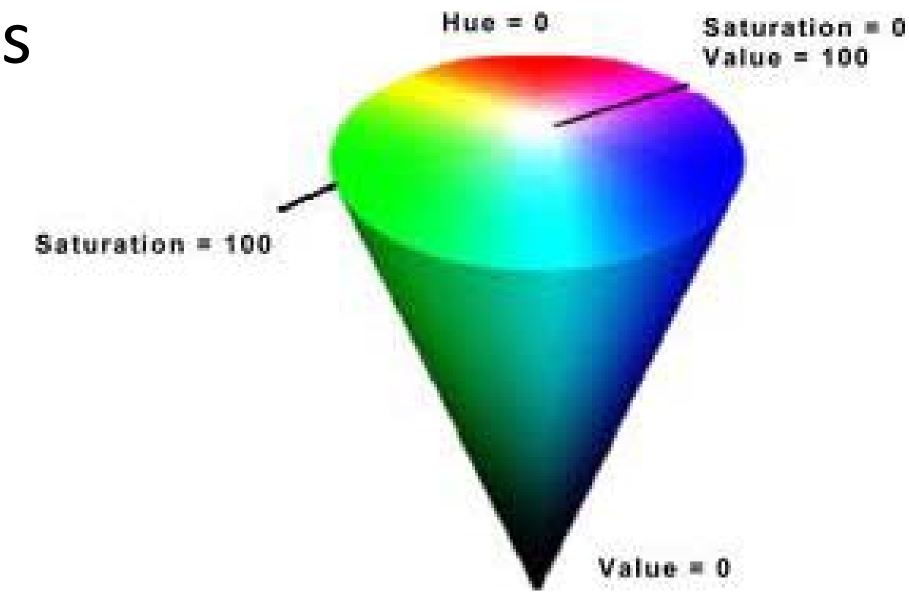
- **The key pt in this talk: any set of things at all can be called the ‘points’ of a ‘space’**
- The best way to endow this ‘space’ with geometry is to define the *distance* $d(P,Q)$ between any pair of points P and Q
- Only a few elementary restrictions (Frechet, 1906):

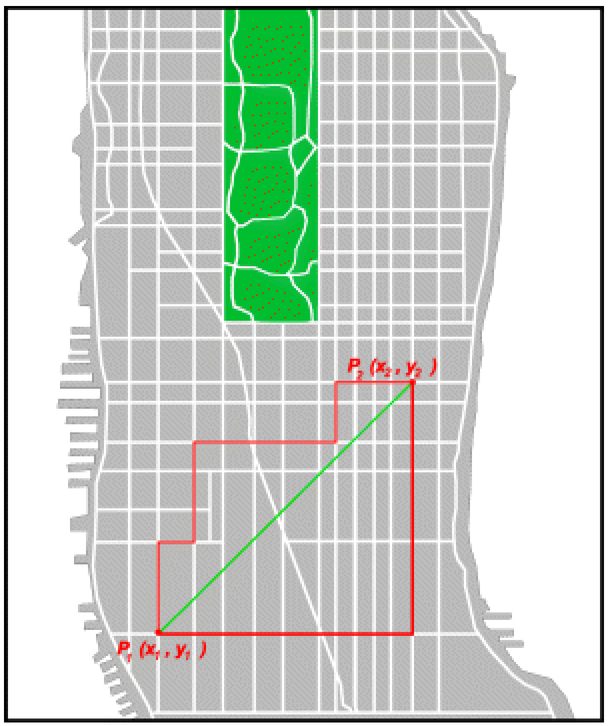
$$d(P,Q) \geq 0, \text{ and } = 0 \text{ only if } P = Q$$

$$d(P,Q) = d(Q,P)$$

$$d(P,R) + d(R,Q) \geq d(P,Q)$$

- You can take odd objects to be "points", for example the set of colors, with distance given by the psychophysical metric

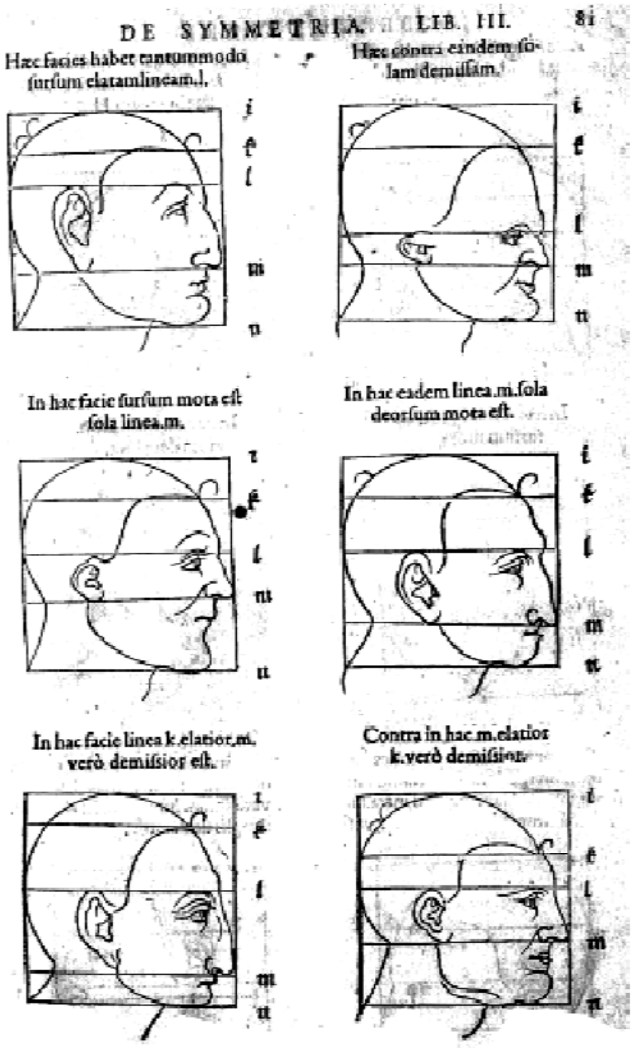


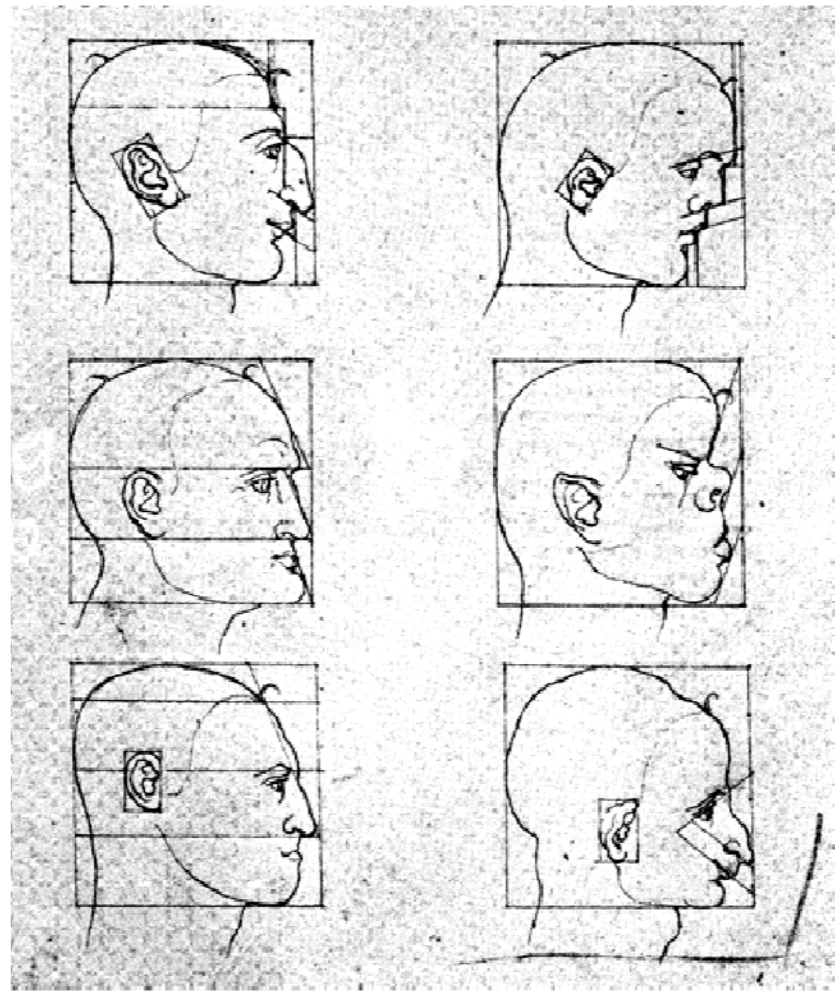


- You can use odd definitions for distance, for example the distance walking along the streets of Manhattan. (Google maps does something like this)

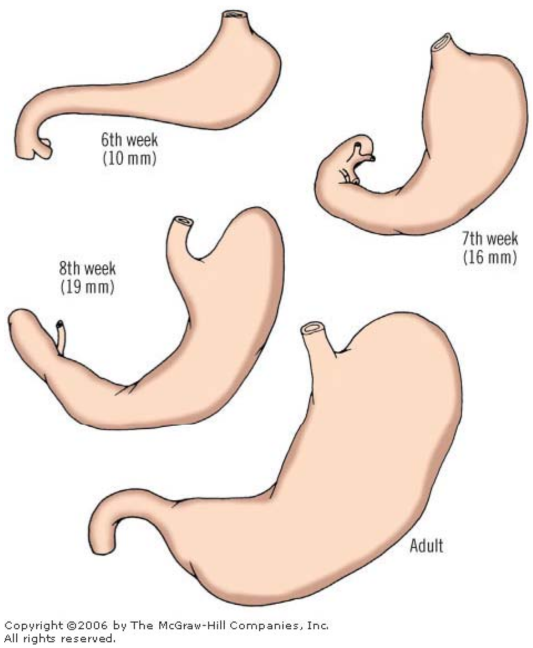
<u>The totality of human faces can be considered as points varying in a space of faces:</u>

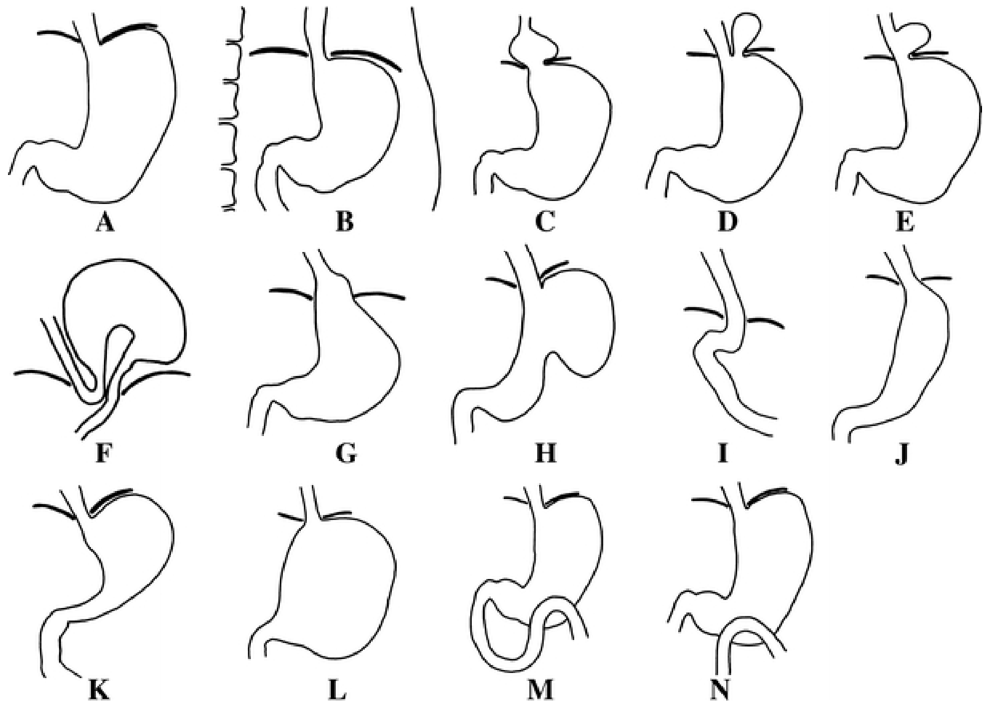
Albrecht Dürer, *Vier Bücher von Menschlicher Proportion,* 1528





Exemplars of some internal organ as points – here the variation in shape of the stomach





<u>Or skulls:</u> D’Arcy Thompson, *On Growth and Form*, 1917. He introduced the idea of warping one skull to the other by simple mathematical functions

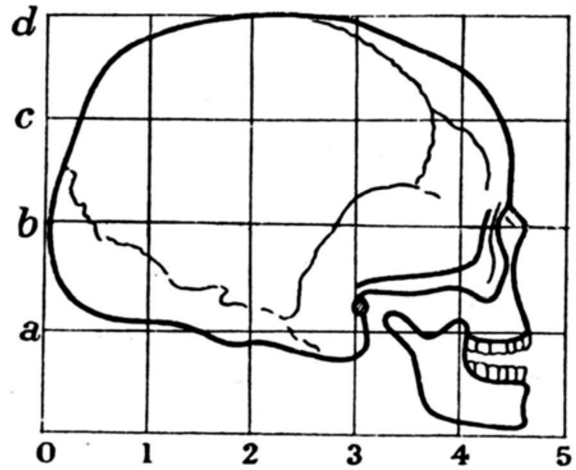


Fig. 177. Human skull.

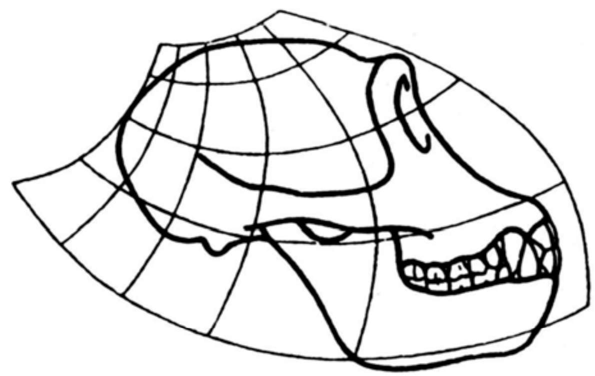


Fig. 179. Skull of chimpanzee.

Fig. 180. Skull of baboon.

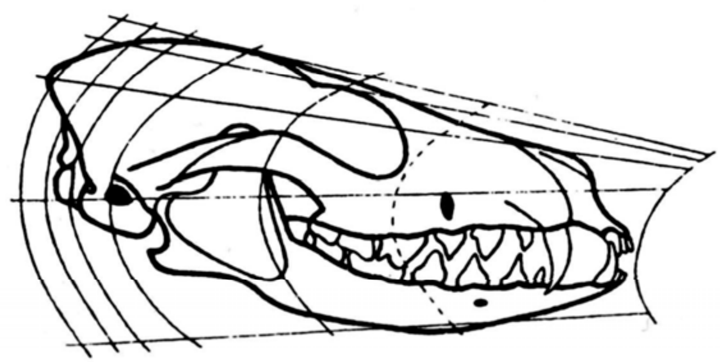


Fig. 181. Skull of dog, compared with the human skull

Thompson’s fish:

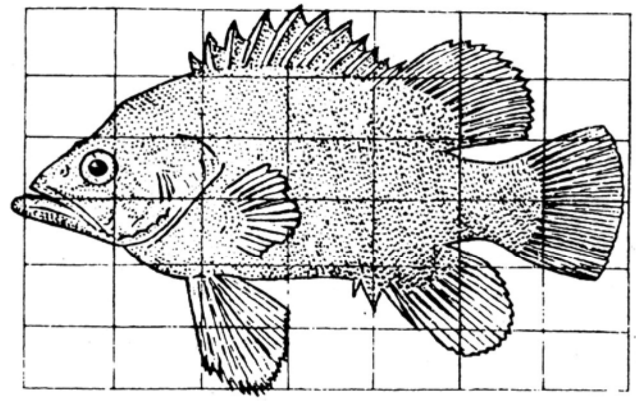


Fig. 150. *Polyprion.*

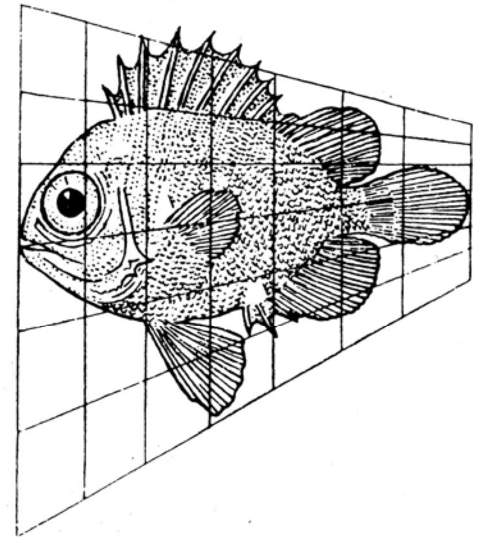


Fig. 151. *Pseudopriacanthus altus.*

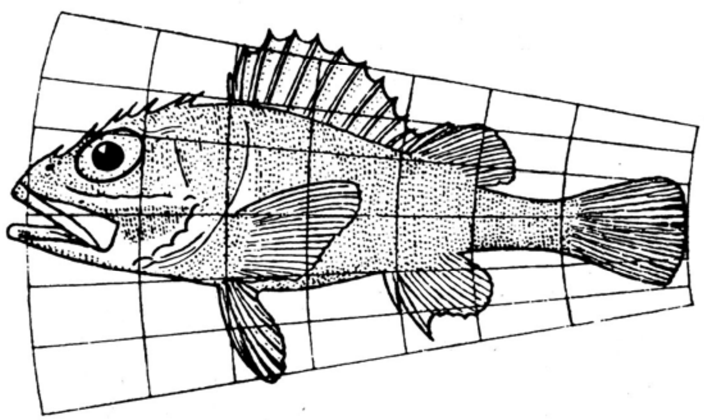


Fig. 152. *Scorpaena* sp.

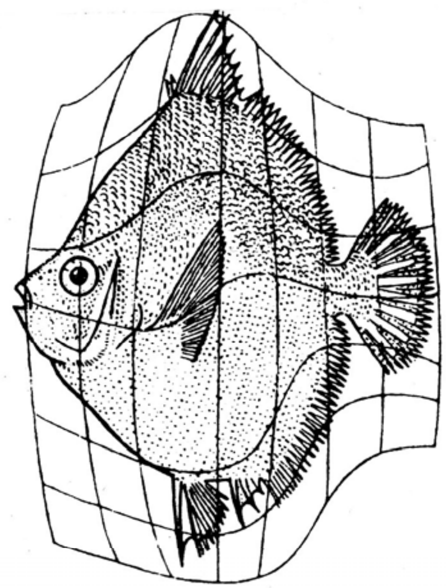
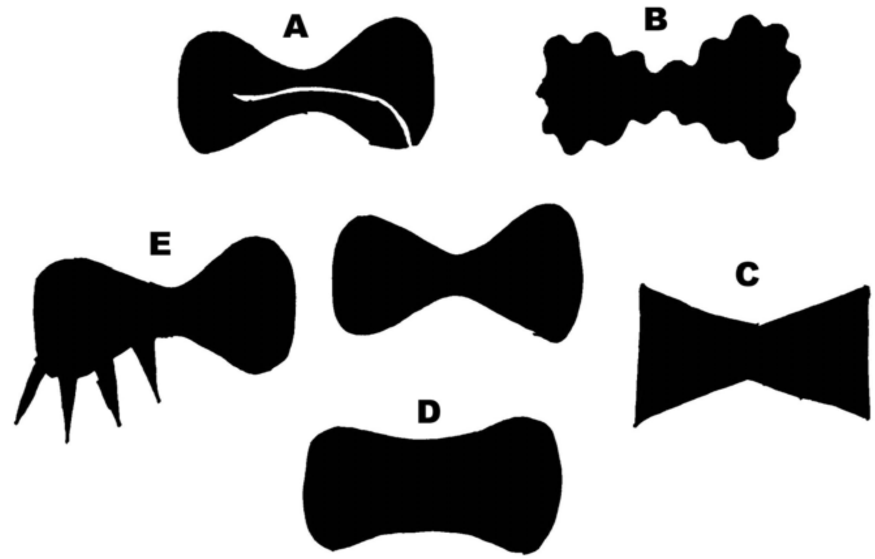


Fig. 153. *Antigonia capros.*

What makes two shapes similar is not simple

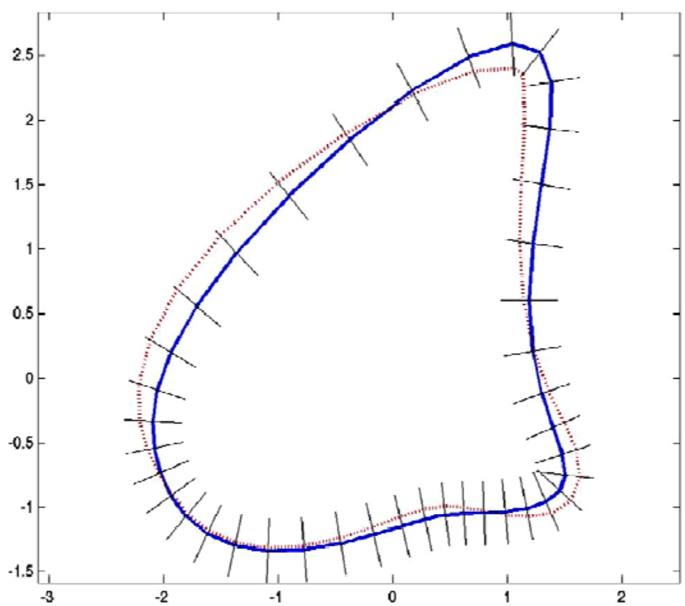
The central shape is similar in various respects to all 5 of the shapes around in – but for different measures of distance!

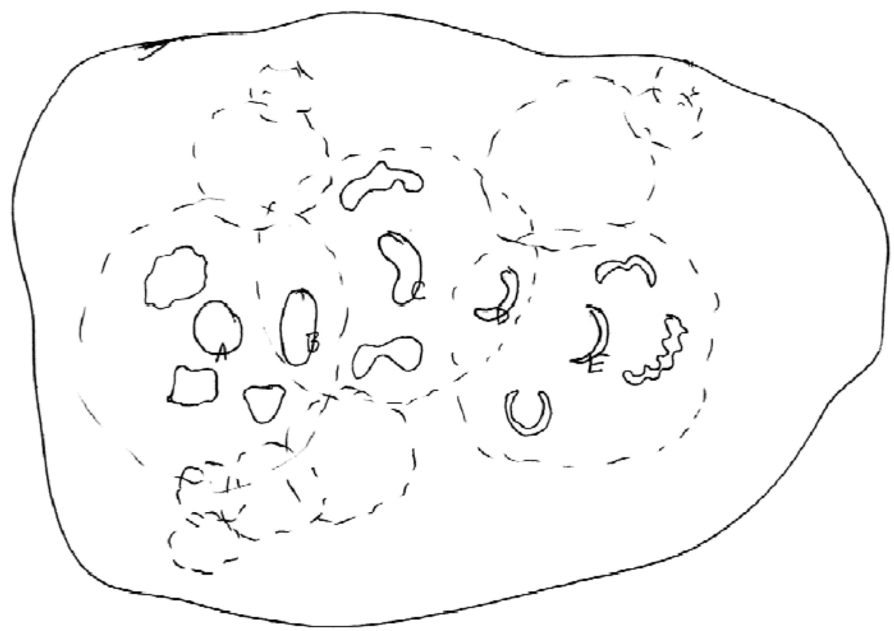


- Looking at area of overlap, distances are: A < B,C < D,E
- Looking at points farthest away, distances are: B < C,D <A,E
- Looking at slopes of boundary: D < B,C < A,E
- Looking at corners: D < A,B < C,E
- Discarding some bad bits, E is identical to the middle
- Thinking of a shape as made of parts, all but D clearly have 2 parts

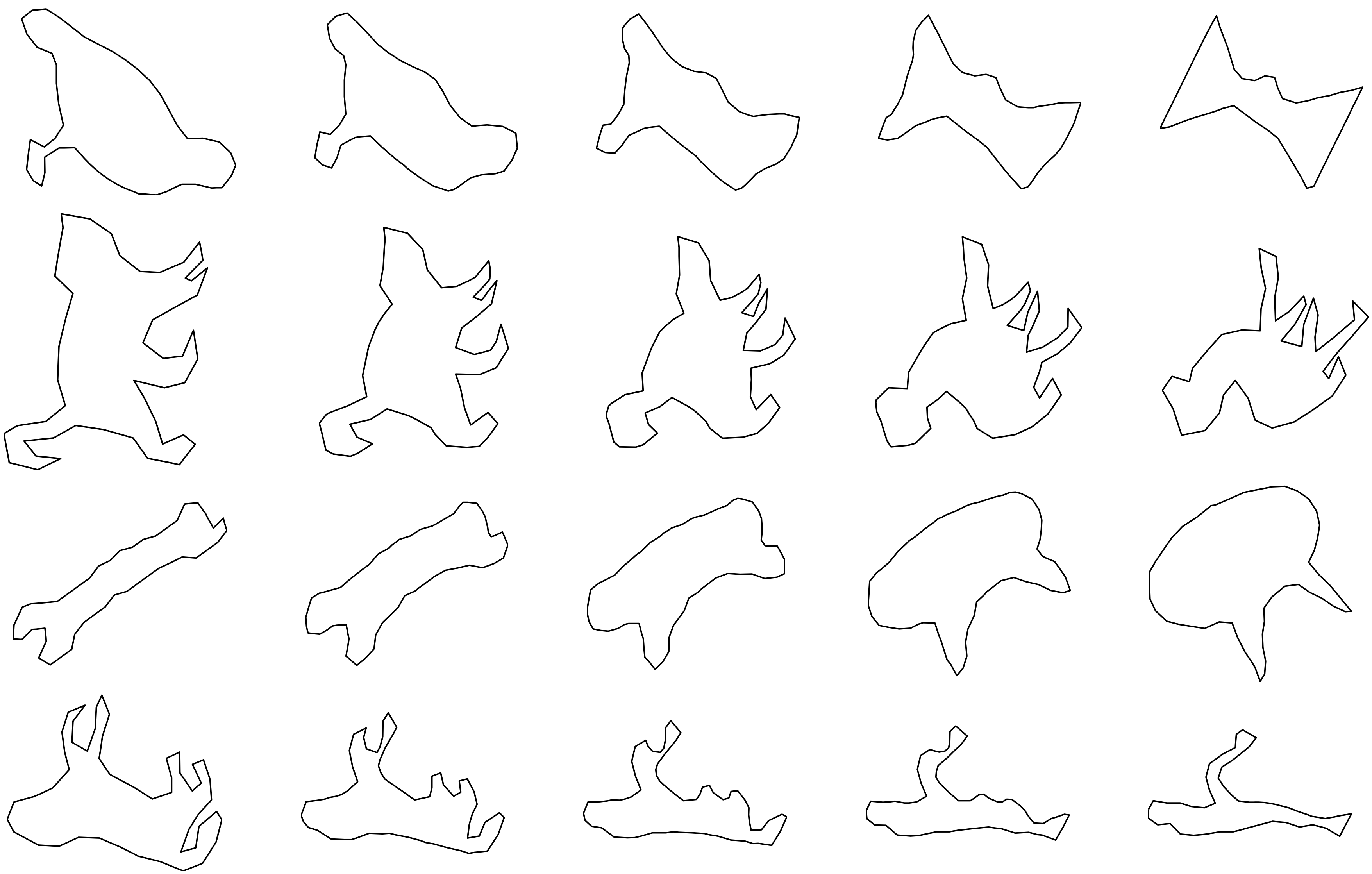
<u>To do math on the space of shapes, you need *coordinates* (Riemann 1854)</u>

You compare nearby shapes by placing 'hairs' on the basic shape and describe the new shape by how far it moves along each hair. To compare very different shapes, you move through multiple such neighborhoods A,B,C,D,E:

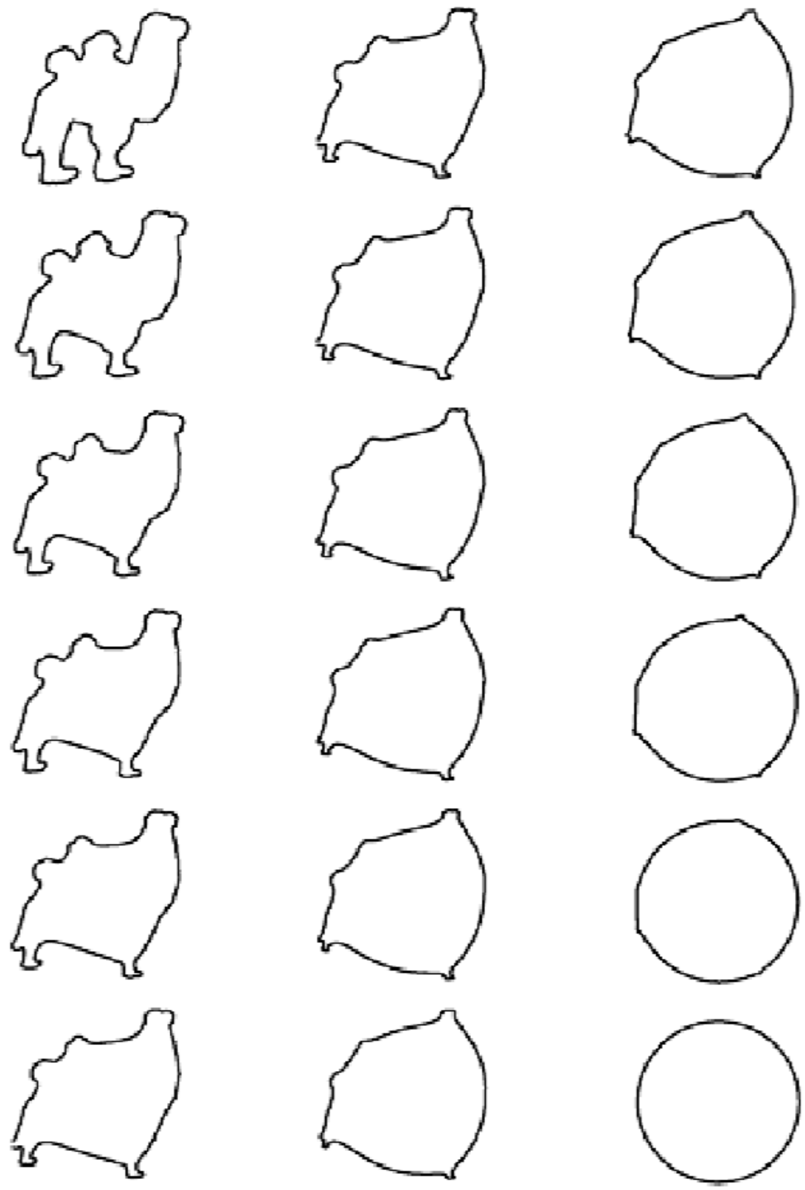




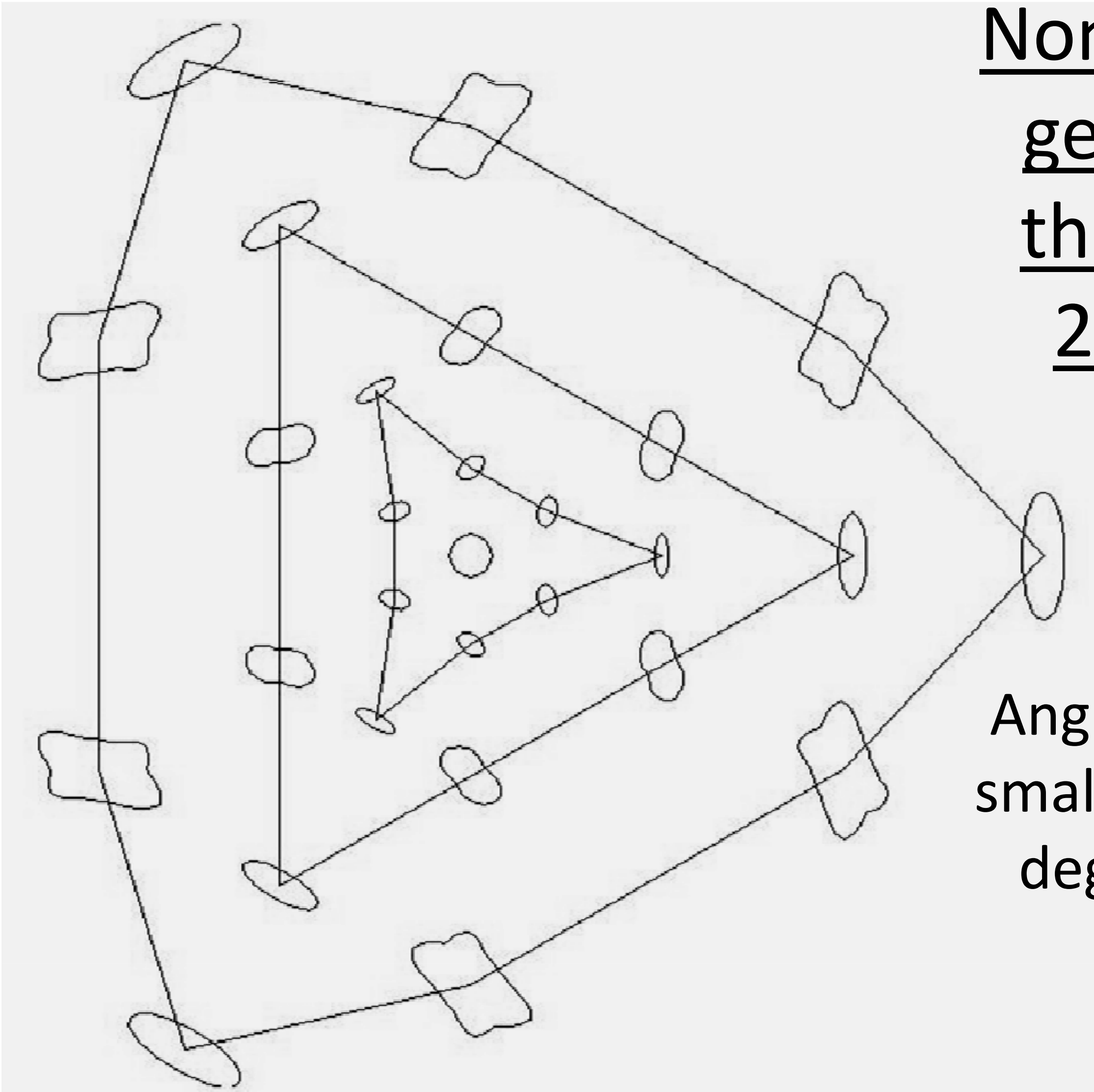
4 examples of geodesics in the space of 2D shapes (Laurent Younes)



A geodesic with a different distance measure (Matt Feiszli)



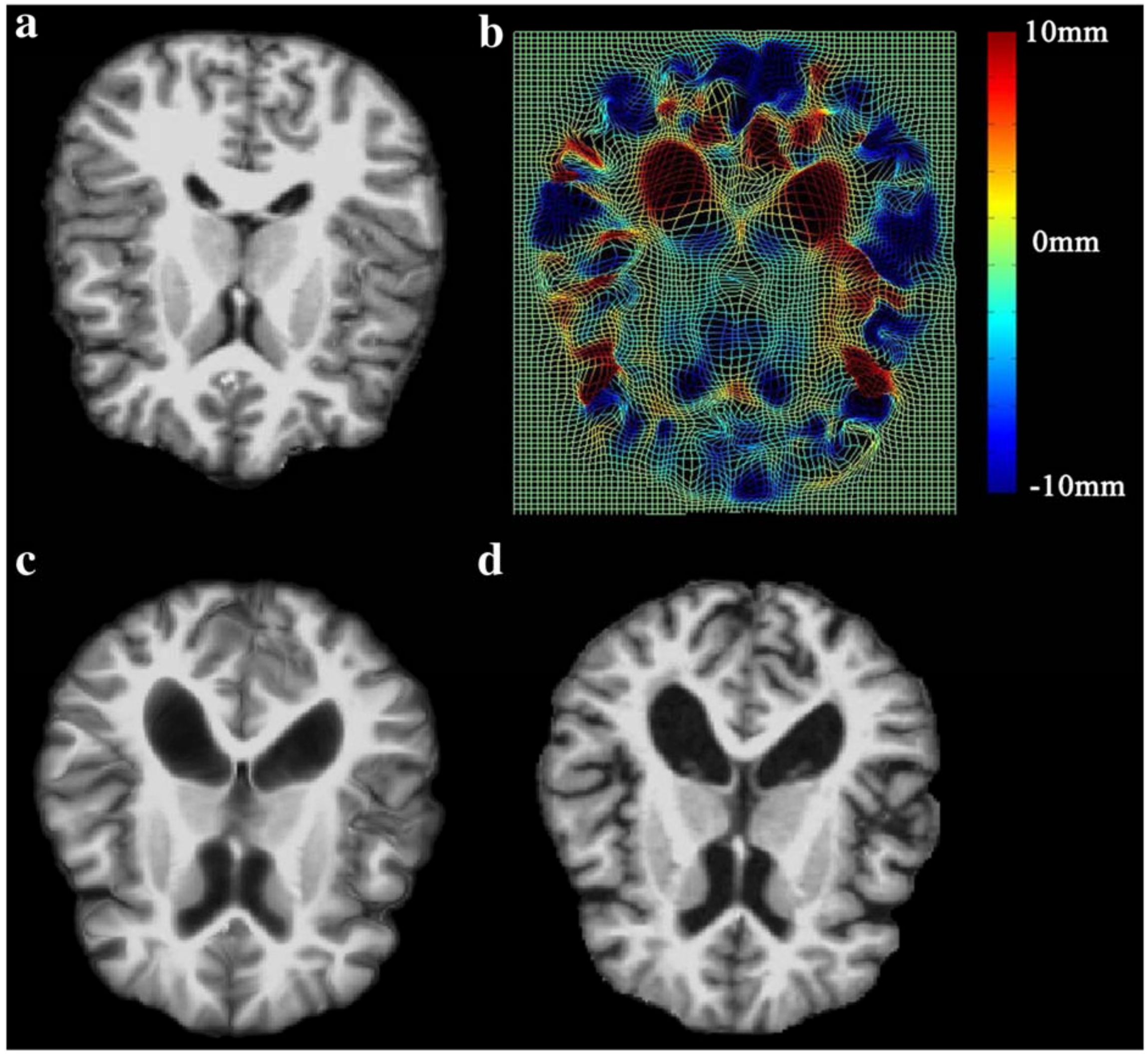
Non-euclidean geometry in the space of 2D shapes



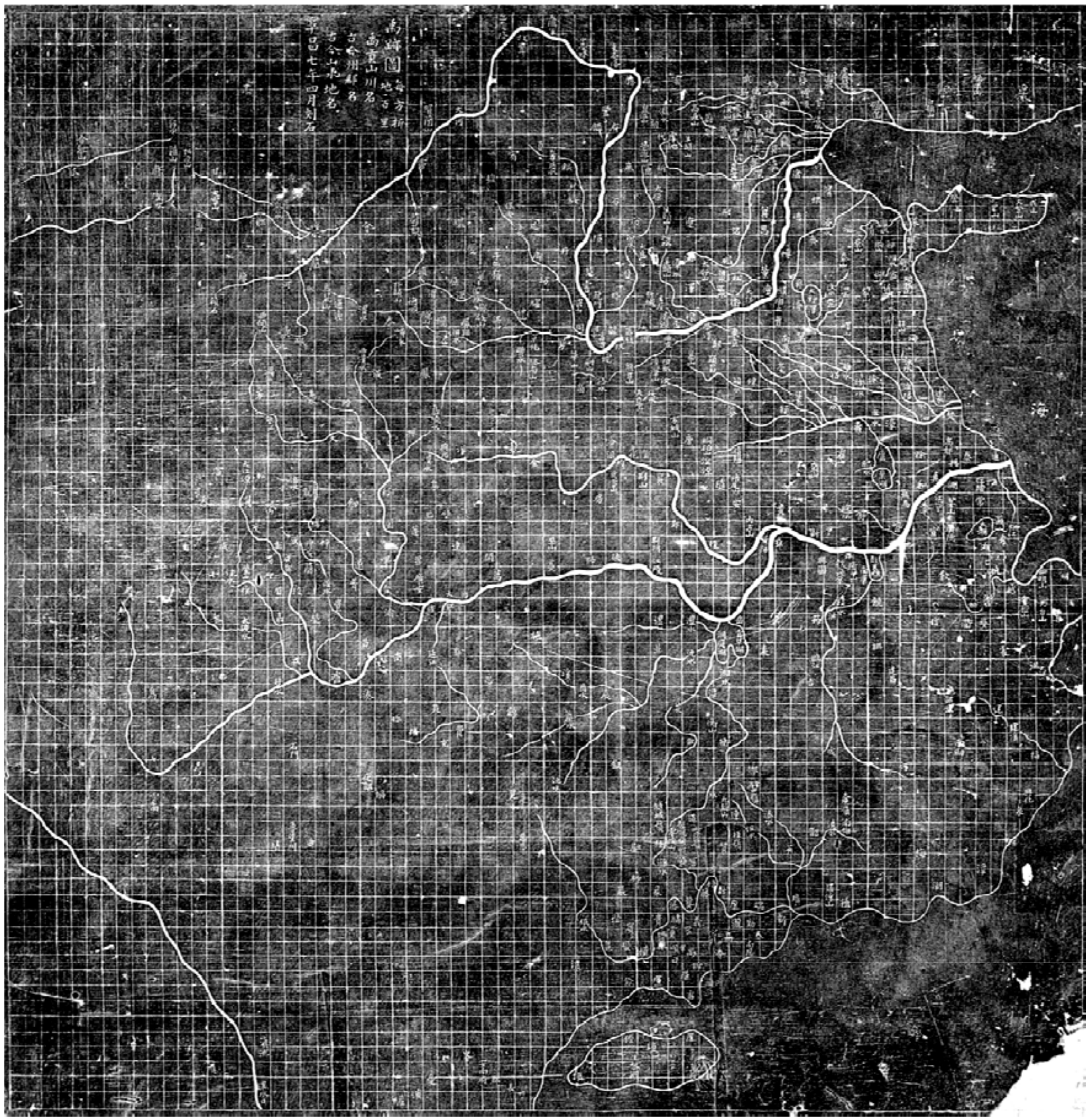
Angle sum 102° in small triangle, 207° degrees in large one

A shape breakthrough in medicine: comparing the full cortex of a patient with a ‘template’ cortex

•Du, Younes, Qiu, *Neuroimage* 2011 *Whole brain diffeomorphic metric mapping*
• a = healthy brain
• d = senile brain with shrunken white matter, enlarged ventricles
• c = warp of a, matching d



<u>The shape of China</u>: *Yu Ji Tu* (Map of the trails of *Yu*)



- Carved in stone, in Xian, 1137 CE, c. 1 meter square
- Note the coastline, the Yellow and Yangtze rivers.
- The grid lines are asserted to have constant spacing 100 *li.*
- However, the north is in Inner Mongolia, the south extends to Hanoi for which the ratio of cosines of latitude is 0.77!

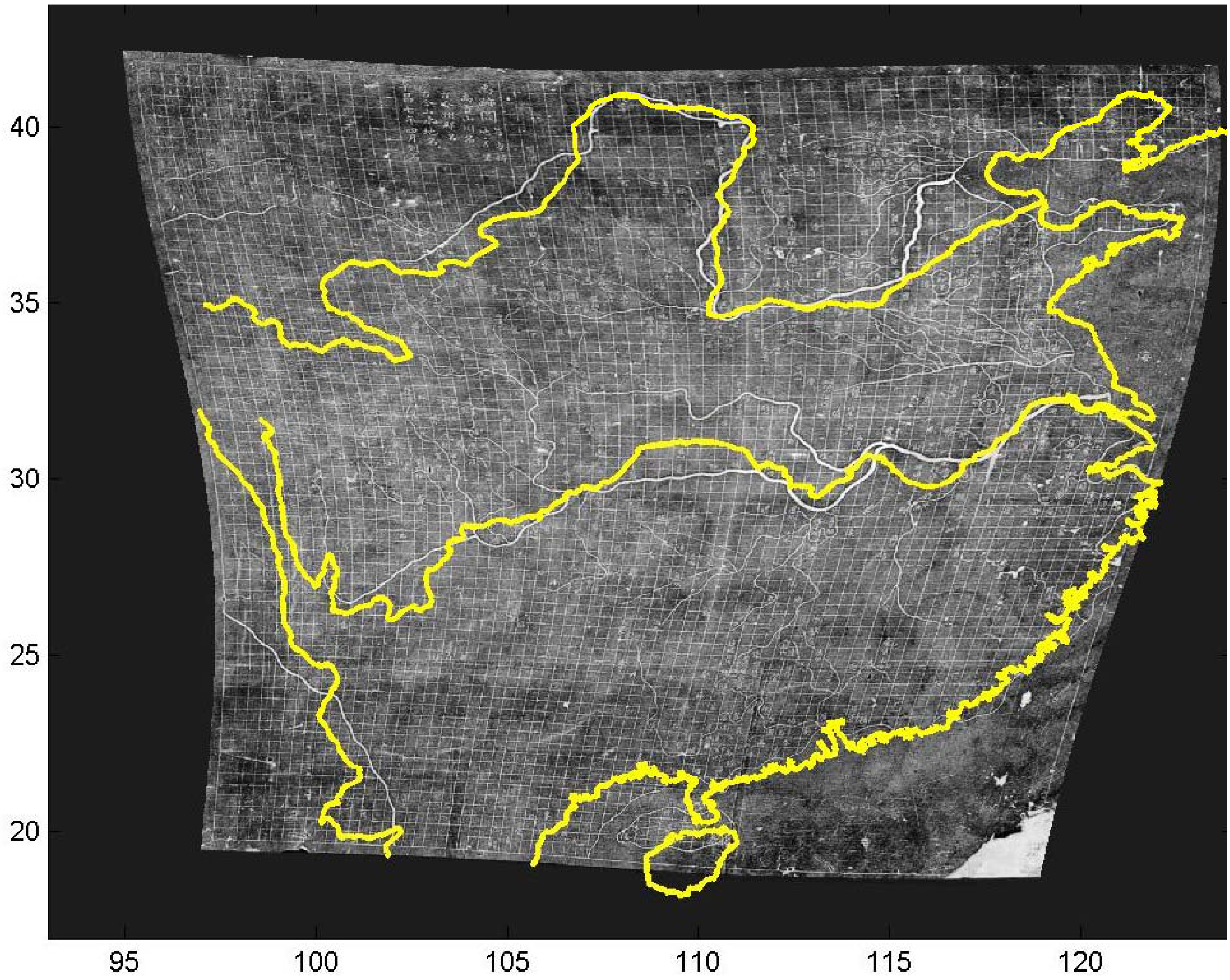
- Chinese culture has an ancient tradition of thinking of the earth as a large flat square
- Cosmogonies with a round earth were discussed but were always in the realm of speculation and were unrealistically huge
- ‘Cartesian’ coordinates were proposed by Pei Xui about 250 CE and maps played a major role in military expeditions.
- Shen Kua was a polymath genius in the Northern Song who *might* have realized this didn’t work. The *Yu Ji Tu* was carved one generation after him and the fall of Northern China to the Jurchen.

Apply D'Arcy Thompson's warping to "georeference" the *Yujitu* to latitude/longitude

Yellow = modern map

White = Yujitu behind it

Note angle of NS grid lines



What have we gained?

- We can consider the set of all colors, all faces, all fish, all human cortices or all maps of some country as points of a space.
- Then we can introduce a measure of distance in this space to quantify differences and find shortest paths warping one shape to another.
- This brings the intuitive but imprecise idea of shape into a mathematical setting for analysis.